

DEGREES BOUNDING PRINCIPLES AND UNIVERSAL INSTANCES IN REVERSE MATHEMATICS

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ABSTRACT. A Turing degree \mathbf{d} *bounds* a principle P of reverse mathematics if every computable instance of P has a \mathbf{d} -computable solution. P admits a *universal instance* if there exists a computable instance such that every solution bounds P . We prove that the stable version of the ascending descending sequence principle (SADS) as well as the stable version of the thin set theorem for pairs (STS(2)) do not admit a bound of low_2 degree. Therefore no principle between Ramsey's theorem for pairs (RT_2^2) and SADS or STS(2) admit a universal instance. We construct a low_2 degree bounding the Erdős Moser theorem (EM), thereby showing that previous argument does not hold for EM. Finally, we prove that the only Δ_2^0 degree bounding a stable version of the rainbow Ramsey theorem for pairs (SRRT_2^2) is $0'$. Hence no principle between the stable Ramsey theorem for pairs (SRT_2^2) and SRRT_2^2 admit a universal instance. In particular the stable version of the Erdős Moser theorem does not admit one. It remains unknown whether EM admits a universal instance.

1. INTRODUCTION

Reverse mathematics is a program whose goal is to classify theorems in function of their computational strength, within the framework of subsystems of second order arithmetic. Proofs are done relatively to a very weak system (RCA_0) meant to capture *computational mathematics*. RCA_0 is composed of basic Peano axioms, Δ_1^0 comprehension and Σ_1^0 induction schemes. See [12] for a good introductory book. Most of statements in reverse mathematics are of the form

$$\forall X(\Phi(X) \rightarrow \exists Y \Psi(X, Y))$$

where Φ and Ψ are arithmetical formulas.

A set X such that $\Phi(X)$ holds is called an *instance* of P and a set Y such that $\Psi(X, Y)$ holds is a *solution* to X . We can see relations between two instances X_1, X_2 of a statement P as a mass problem consisting of computing a solution to X_1 given any solution to X_2 .

Definition 1.1 Given a statement P , a degree \mathbf{d} is *P-bounding* ($\mathbf{d} \gg_P \emptyset$) if every computable instance X of P has a \mathbf{d} -computable solution. A statement P admits a *universal instance* if it has a computable instance X such that every solution to X bounds P .

The notation $\mathbf{d} \gg \emptyset$ historically means that the degree \mathbf{d} is PA and therefore is equivalent to $\mathbf{d} \gg_{\text{WKL}_0} \emptyset$ where WKL_0 is the weak König's lemma principle, i.e. König's lemma restricted to subtrees of $2^{<\omega}$. It is well-known that WKL_0 admits a universal instance – e.g. take the Π_1^0 class of completions of Peano arithmetics –. A few principles have been proven to admit universal instances – WKL_0 [22], König's lemma (KL)

[12], the Ramsey-type weak König's lemma (RWWKL) [1], the finite intersection property (FIP) [9], the omitting partial type theorem (OPT) [15], or even the rainbow Ramsey theorem for pairs (RRT_2^2) [21] – but most of principles do not admit one. An important notion for proving such a result is computable reducibility.

Definition 1.2 A statement P is *computably reducible* to a statement Q (written $P \leq_c Q$) if for every instance X of P there exists an instance Y of Q computable from X such that each solution to Y computes relative to X a solution to X .

Mileti proved in [20] that the stable Ramsey theorem for pairs (SRT_2^2) admits no bound of low_2 degree. Therefore every statement P having an ω -model with only low_2 sets, and such that $\text{SRT}_2^2 \leq_c P$, admits no universal instance. In particular none of Ramsey's theorem for pairs (RT_2^2), SRT_2^2 and the Ramsey-type weak König's lemma relative to \emptyset' ($\text{RWKL}[\emptyset']$) admit a universal instance. Independently, Hirschfeldt & Shore proved in [14] that the stable ascending descending sequence principle (SADS) admits no bound of low degree. Hence none of SADS and the stable chain antichain principle (SCAC) admit a universal instance.

We generalize both results by proving that SADS admits not bound of low_2 degree, proving therefore that if a statement P has an ω -model with only low_2 sets and $\text{SADS} \leq_c P$ then P admits no universal instance. We also extend the result to statements to which the stable thin set theorem for pairs ($\text{STS}(2)$) computably reduces. Hence we deduce that none of the ascending descending sequence principle (ADS), the chain antichain principle (CAC), the thin set theorem for pairs ($\text{TS}(2)$), the free set theorem for pairs ($\text{FS}(2)$) and their stable versions admit a universal instance.

We generalize the result to arbitrary tuples and prove that none of RT_2^n , $\text{FS}(n)$, $\text{TS}(n)$ and their stable versions admit a universal instance for $n \geq 2$. The question remains open for the rainbow Ramsey theorem for n -tuples (RRT_2^n) with $n \geq 3$. We construct a low_2 degree bounding the Erdős Moser theorem (EM), thereby showing that previous argument does not hold for EM.

Mileti proved in [20] that the only Δ_2^0 degree bounding SRT_2^2 is \emptyset' . Using the fact that every Δ_2^0 set has an infinite incomplete Δ_2^0 subset in either it or its complement [13], we obtain another proof that SRT_2^2 admits no universal instance. We extend this result by proving that the only Δ_2^0 degree bounding a stable version version of the rainbow Ramsey theorem for pairs (SRRT_2^2) is \emptyset' . Hence none of the statements P satisfying $\text{SRRT}_2^2 \leq_c P \leq_c \text{SRT}_2^2$ admit a universal instance. In particular we deduce that neither SRRT_2^2 nor the stable version of the Erdős Moser theorem (SEM) admits a universal instance.

1.1. Notations. Formulas. The notation $(\forall^\infty s)\varphi(s)$ means that $\varphi(s)$ holds for all but finitely many s , i.e. is translated to $(\exists s_0)(\forall s \geq s_0)\varphi(s)$. Given two sets X and Y , we denote by $X \subseteq^* Y$ the statement $(\forall^\infty s \in X)[s \in Y]$. Accordingly, $X =^* Y$ means that both $X \subseteq^* Y$ and $Y \subseteq^* X$ hold, i.e. X and Y differ by finitely many elements.

Turing functional and lowness. We fix an effective enumeration of all Turing functionals Φ_0, Φ_1, \dots . We denote by $\Phi_{e,s}$ the partial approximation of the Turing functional Φ_e at stage s . Given a set X , we denote by X' the jump of X and by $X^{(n)}$ the n th jump of X . A set X is low_n over Y if $(X \oplus Y)^{(n)} \leq Y^{(n)}$. A set is low_n if it is low_n over \emptyset . A low_n -ness index of a set X low_n over Y is a Turing index e such that $\Phi_e^{Y^{(n)}} = (X \oplus Y)^{(n)}$.

Mathias forcing. Given two sets E and F , we denote by $E < F$ the formula $(\forall x \in E)(\forall y \in F)x < y$. A *Mathias condition* is a pair (F, X) where F is a finite set, X is an infinite set and $F < X$. A condition (\tilde{F}, \tilde{X}) *extends* (F, X) (written $(\tilde{F}, \tilde{X}) \leq (F, X)$) if $F \subseteq \tilde{F}$, $\tilde{X} \subseteq X$ and $\tilde{F} \setminus F \subset X$. A set G *satisfies* a Mathias condition (F, X) if $F \subset G$ and $G \setminus F \subseteq X$.

2. DEGREES BOUNDING COHESIVENESS

A standard proof of Ramsey's theorem for pairs consists of reducing an arbitrary coloring of pairs into a *stable* one using the cohesiveness principle. The understanding of the links between cohesiveness and stability is a very active subject of research in reverse mathematics [4, 13, 5].

Definition 2.1 (Cohesiveness) An infinite set C is \vec{R} -cohesive for a sequence of sets R_0, R_1, \dots if for each $i \in \omega$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$. A set C is *cohesive* (resp. *r-cohesive*) if it is \vec{R} -cohesive where \vec{R} is an enumeration of all c.e. (resp. computable) sets. COH is the statement “Every uniform sequence of sets \vec{R} has an \vec{R} -cohesive set.”

Jockusch & al. proved in [16] the existence of a low_2 cohesive set. Degrees bounding COH are quite well understood and admit a simple characterization:

Theorem 2.2 (Jockusch & Stephan [16]) Fix an $n \in \omega$.

1. For every set C such that $C' \gg \emptyset'$, $C \gg_{\text{COH}} \emptyset$.
2. There exists a uniformly $\emptyset^{(n)}$ -computable sequence of sets \vec{R} such that for every \vec{R} -cohesive set C , $(C \oplus \emptyset^{(n)})' \gg \emptyset^{(n+1)}$.

In particular, taking a set $P \gg \emptyset'$ low over \emptyset' and a set C such that $C' =_T P$ whose existence is ensured by Friedberg's jump inversion theorem, we obtain a low_2 degree bounding COH. The canonical $\emptyset^{(n)}$ -computable sequence of sets \vec{R} whose existence is claimed in clause 2 of Theorem 2.2 is

$$R_e = \{s : \Phi_{e,s}^{\emptyset^{(n+1)}}(e) \downarrow = 1\}$$

Every \vec{R} -cohesive set C computes a function $f(\cdot, \cdot)$ such that $\lim_{s \in C} f(e, s)$ exists for each $e \in \omega$ and $\lim_{s \in C} f(e, s) = \Phi_e^{\emptyset^{(n+1)}}(e)$ for each Turing index e such that $\Phi_e^{\emptyset^{(n+1)}}(e) \downarrow$. By a relativized version of Schoenfield's limit lemma, $(C \oplus \emptyset^{(n)})'$ computes the function $\tilde{f}(x) = \lim_{s \in C} f(x, s)$ and is therefore of PA degree relative to $\emptyset^{(n+1)}$.

Corollary 2.3 COH admits a universal instance.

Proof. The uniformly computable sequence of sets \vec{R} such that the jump of every \vec{R} -cohesive set is of PA degree relative to \emptyset' is a universal instance by previous theorem. \square

Wang proved in [26] that for every set $P \gg \emptyset''$ and every uniformly \emptyset' -computable sequence of sets \vec{R} , there exists an \vec{R} -cohesive set C such that $C'' \leq_T C \oplus \emptyset'' \leq_T P$. Cholak & al. used in [4] the existence of a low subuniform degree to deduce the existence, for every set $P \gg \emptyset'$, of an r-cohesive set C such that $C' \leq_T P$. We can apply a similar reasoning for \emptyset' -computable sets, using the fact that degrees bounding COH are somehow subuniform degrees for Δ_2^0 approximations.

Theorem 2.4 For every set $P \gg \emptyset''$, there exists an \vec{R} -cohesive set C such that $C'' \leq_T C \oplus \emptyset'' \leq_T P$, where \vec{R} is the (non-uniformly computable) sequence of all \emptyset' -computable sets.

Proof. Let \vec{U} be the uniformly computable sequence of sets defined by

$$U_{e,x} = \{s : \Phi_{e,s}^{\emptyset'}(x) = 1\}$$

Fix a low₂ \vec{U} -cohesive set C_0 and its C_0 -computable bijection $f : \omega \rightarrow C_0$. Every set $P \gg \emptyset''$, $P \gg C_0''$. Consider the uniformly C_0' -computable sequence of sets

$$V_e = \{x : \lim_s \Phi_{e,s}^{\emptyset'_{f(s)}}(x) = 1\}$$

The sequence \vec{V} contains every \emptyset' -computable set. In particular, every \vec{V} -cohesive set is \vec{R} -cohesive. By a relativization of Wang's result, there exists an \vec{V} -cohesive set C such that $(C \oplus C_0)'' \leq_T C \oplus C_0'' =_T C \oplus \emptyset'' \leq_T P$. \square

The proof of previous theorem shows that an application of COH followed by an application of COH[\emptyset'] are enough to obtain a set of degree bounding COH[\emptyset']. The following question remains open:

Question 2.5 Does COH[\emptyset'] admit a universal instance ?

3. DEGREES BOUNDING THE ATOMIC MODEL THEOREM

The atomic model theorem is a statement of model theory admitting a simple, purely computability theoretic characterization over ω -models. This statement happens to have a weak computational content and is therefore a consequence of many other principles in reverse mathematics. For those reasons, the atomic model theorem is a good candidate for factorizing proofs of properties which are closed upward by the consequence relation.

Definition 3.1 (Atomic model theorem) A formula $\varphi(x_1, \dots, x_n)$ of T is an *atom* of a theory T if for each formula $\psi(x_1, \dots, x_n)$, one of $T \vdash \varphi \rightarrow \psi$ and $T \vdash \varphi \rightarrow \neg\psi$ holds, but not both. A theory T is *atomic* if, for every formula $\psi(x_1, \dots, x_n)$ consistent with T , there exists an atom $\varphi(x_1, \dots, x_n)$ of T extending it, i.e. one such that $T \vdash \varphi \rightarrow \psi$. A model \mathcal{A} of T is *atomic* if every n -tuple from \mathcal{A} satisfies an atom of T . AMT is the statement “Every complete atomic theory has an atomic model”.

AMT has been introduced as a principle by Hirschfeldt & al. in [15]. They proved that WKL₀ and AMT are incomparable on ω -models, proved over RCA₀ that AMT is strictly weaker than SADS. The author proved in [23] that STS(2) implies AMT over RCA₀. In this section we use the fact that AMT is not bounded by any Δ_2^0 low₂ degree to deduce that none of AMT, SADS and SCAC admits a universal instance. The principle AMT has been proven in [15, 6] to be computably equivalent to the following principle:

Definition 3.2 (Escape property) For every Δ_2^0 function f , there exists a function g such that $f(x) \leq g(x)$ for infinitely many x .

This equivalence does not hold over RCA_0 as, unlike AMT, the escape property implies $\text{I}\Sigma_2^0$ over $\text{B}\Sigma_2^0$ [15]. Using this characterization, we can easily deduce the two following theorems:

Theorem 3.3 (Hirschfeldt & al. [15]) There is no $\text{low}_2 \Delta_2^0$ degree bounding AMT.

Theorem 3.4 No principle P having an ω -model with only low sets and such that $\text{AMT} \leq_c P$ admits a universal instance.

Theorem 3.3 and Theorem 3.4 can be easily proven using the following characterization of $\Delta_2^0 \text{low}_2$ sets in terms of domination:

Lemma 3.5 (Martin, [19]) A set $A \leq_T \emptyset'$ is low_2 iff there exists an $f \leq_T \emptyset'$ dominating every A -computable function.

Proof. A set A is low_2 iff \emptyset' is high relative to A . As a set X is high relative to a set $A \leq_T \emptyset'$ iff it computes a function dominating every A -computable function, we conclude. \square

Remark. As explained Conidis in [6], Theorem 3.3 cannot be extended to every low_2 sets: Soare [6] constructed a low_2 set bounding the escape property using a forcing argument. So there exists a low_2 degree bounding AMT.

Proof of Theorem 3.4. Suppose for the sake of contradiction that P has a universal instance U and an ω -model \mathcal{M} with only low sets. As U is computable, $U \in \mathcal{M}$. Let $X \in \mathcal{M}$ be a (low) solution to U . In particular, X is low_2 and Δ_2^0 , so by Lemma 3.5 and the computable equivalence of AMT and the escape property, there exists a computable instance Y of AMT such that X does not compute a solution to Y . As $\text{AMT} \leq_c P$, there exists a Y -computable (hence computable) instance Z of P such that every solution to Z computes a solution to Y . Thus X does not compute a solution to Z , contradicting universality of U . \square

Hirschfeldt & al. proved in [14] the existence of an ω -model of SADS and SCAC with only low sets. Therefore we obtain another proof that neither SADS nor SCAC admits a universal instance. The result was first proven in [14] using an ad-hoc notion of reducibility.

Corollary 3.6 None of AMT, SADS and SCAC admit a universal instance.

Previous argument can not directly be applied to SRT_2^2 , SEM or $\text{STS}(2)$ as none of those principles admit an ω -model with only low sets [10, 17, 23]. However Lemma 3.4 can be extended to principles such that every computable instance has a $\Delta_2^0 \text{low}_2$ solution. It is currently unknown whether every Δ_2^0 set admits a $\Delta_2^0 \text{low}_2$ infinite subset in either it or its complement. A positive answer would lead to a proof that SRT_2^2 , SEM and $\text{STS}(2)$ have no universal instance, and more importantly, would provide an ω -model of SRT_2^2 not model of $\text{DNR}[\emptyset']$ as explained in [13]. We shall see later that none of SRT_2^2 , SEM and $\text{STS}(2)$ admits a universal instance.

4. DEGREES BOUNDING $\text{STS}(2)$ AND SADS

Mileti originally proved in [20] that no principle P having an ω -model with only low_2 sets and satisfying $\text{SRT}_2^2 \leq_c P$ admits a universal instance, and deduced that none

of SRT_2^2 and RT_2^2 admit one. In this section, we reapply his argument to much weaker statements and derive non-universality results to a large range of principles in reverse mathematics. Thin set theorem and ascending descending sequence are example of statements weak enough to be a consequence of many others, and surprisingly strong enough to diagonalize against low_2 sets.

Definition 4.1 (Thin set) Let $k \in \omega$ and $f : [\omega]^k \rightarrow \omega$. A set A is *thin for f* if $f([A]^n) \neq \omega$, that is, if the set A “avoids” at least one color. $\text{TS}(k)$ is the statement “every function $f : [\omega]^k \rightarrow \omega$ has an infinite set thin for f ”. $\text{STS}(k)$ is the restriction of $\text{TS}(k)$ to stable functions.

Cholak & al. studied extensively thin set principle in [3]. Some of the results were already stated by Friedman without giving a proof, notably there exists an ω -model of WKL_0 which is not a model of $\text{TS}(2)$, and the arithmetical comprehension axiom (ACA_0) does not imply $(\forall k)\text{TS}(k)$ over RCA_0 . Wang showed in [27] that $(\forall k)\text{TS}(k)$ does not imply ACA_0 on ω -models. Rice [24] proved that $\text{STS}(2)$ implies DNR over RCA_0 . The author proved in [23] that $\text{RCA}_0 \vdash \text{TS}(2) \rightarrow \text{RRT}_2^2$.

Definition 4.2 (Ascending descending sequence) ADS is the statement “Every linear order admits an infinite ascending or descending sequence”. SADS is the restriction of ADS to order types $\omega + \omega^*$.

Tennenbaum [25] constructed a computable linear order of order type $\omega + \omega^*$ with no computable ascending or descending sequence. Therefore SADS does not hold over RCA_0 . Hirschfeldt & Shore [14] studied ADS within the framework of reverse mathematics, proving that ADS imply both COH and BS_2^0 over RCA_0 and that SADS implies AMT over RCA_0 . They constructed an ω -model of ADS not model of DNR , and an ω -model of $\text{COH} + \text{WKL}_0$ not model of SADS .

The study of degrees bounding a statement and the existence of a universal instance are closely related. As does Mileti in [20], we deduce two kind of theorems by the application of his proof technique.

Theorem 4.3 There exists no low_2 degree bounding any of $\text{STS}(2)$ or SADS .

Theorem 4.4 No principle P having an ω -model with only low_2 sets and such that any of $\text{STS}(2)$, SADS is computably reducible to P admits a universal instance.

The proof of the two theorems is split into three lemmas. Lemma 4.7 provides a general way of obtaining bounding and universality results, assuming the ability of a principle to diagonalize against a particular set. Lemma 4.8 and Lemma 4.9 state the desired diagonalization for respectively $\text{STS}(2)$ and SADS .

Corollary 4.5 None of the following principles admits a universal instance: RT_2^2 , $\text{RWKL}[\emptyset']$, $\text{FS}(2)$, $\text{TS}(2)$, CAC , ADS and their stable versions.

Proof. Each of the above mentioned principles is a consequence of RT_2^2 over RCA_0 and computably implies either SADS or $\text{STS}(2)$. See [11] for $\text{RWKL}[\emptyset']$, [3] for $\text{FS}(2)$ and $\text{TS}(2)$, and [14] for CAC and ADS . By Theorem 3.1 of [4], there exists an ω -model of RT_2^2 having only low_2 sets. We conclude by Theorem 4.4. \square

In order to prove Theorem 4.3 and Theorem 4.4, we need the following theorem proven by Mileti. It simply consists of applying a relativized version of the low basis theorem to a Π_1^0 class of completions of the enumeration of all partial computable sets.

Theorem 4.6 (Mileti, Corollary 5.4.5 of [20]) For every set X , there exists $f : \omega^2 \rightarrow \{0, 1\}$ low over X such that for every X -computable set Z , there exists an $e \in \omega$ with $Z = \{a \in \omega : f(e, a) = 1\}$.

Lemma 4.7 Fix an $n \in \omega$ and two principles P and Q such that $P \leq_c Q$. Suppose that for any $f : \omega^2 \rightarrow \{0, 1\}$ satisfying $f'' \leq_T \emptyset^{(n+2)}$, there exists a computable instance I of P such that for each $e \in \omega$, if $\{a \in \omega : f(e, a) = 1\}$ is infinite then it is not a solution to I . Then the following holds:

- (i) For any degree \mathbf{d} low₂ over $\emptyset^{(n)}$ there is a computable instance U of P such that \mathbf{d} does not bound a solution to U .
- (ii) There is no degree low₂ over $\emptyset^{(n)}$ bounding P .
- (iii) If every computable instance I of Q has a solution low₂ over $\emptyset^{(n)}$, then Q has no universal instance.

Proof.

- (i) Consider any set X of degree low₂ over $\emptyset^{(n)}$. By Theorem 4.6, there exists a function $f : \omega^2 \rightarrow \{0, 1\}$ low over X , hence low₂ over $\emptyset^{(n)}$, such that any X -computable set Z is of the form $\{a \in \omega : f(e, a) = 1\}$ for some $e \in \omega$. Take a computable instance I of P having no solution of the form $\{a \in \omega : f(e, a) = 1\}$ for any $e \in \omega$. Then X does not compute a solution to I .
- (ii) Immediate from (i).
- (iii) Take any computable instance U of Q . By assumption, U has a solution X low₂ over $\emptyset^{(n)}$. By (i), there exists an instance I of P such that X does not compute a solution to I . As $P \leq_c Q$, there exists an I -computable (hence computable) instance J of Q such that any solution to J computes a solution to I . Then X does not compute a solution to J , hence U is not a universal instance.

□

We will prove the following lemmas which, together with Lemma 4.7, are sufficient to deduce Theorem 4.3 and Theorem 4.4.

Lemma 4.8 Fix a set X . Suppose $f : \omega^2 \rightarrow \{0, 1\}$ satisfies $f'' \leq_T X''$. There exists an X -computable stable coloring $g : [\omega] \rightarrow \omega$ such that for all $e \in \omega$, if $\{a \in \omega : f(e, a) = 1\}$ is infinite then it is not thin for g .

Lemma 4.9 Fix a set X . Suppose $f : \omega^2 \rightarrow \{0, 1\}$ satisfies $f'' \leq_T X''$. There exists a stable X -computable linear order L such that for all $e \in \omega$, if $\{a \in \omega : f(e, a) = 1\}$ is infinite then it is neither an ascending nor a descending sequence in L .

Before proving the two remaining lemmas, we relativize the results to colorings over arbitrary tuples.

Theorem 4.10 For any n , there exists no degree low₂ over $\emptyset^{(n)}$ bounding $\text{STS}(n+2)$.

Proof. Apply Lemma 4.8 relativized to $X = \emptyset^{(n)}$ together with Lemma 4.7. Simply notice that if $f : [\omega]^n \rightarrow \omega$ is a \emptyset' -computable coloring, the computable coloring $g : [\omega]^{n+1} \rightarrow \omega$ obtained by an application of Schoenfield's limit lemma is such that every infinite set thin for g is thin for f . \square

Theorem 4.11 For any n , no principle P having an ω -model with only low_2 over $\emptyset^{(n)}$ sets and such that $\text{STS}(n+2) \leq_c P$ admits a universal instance.

Proof. Same reasoning as Theorem 4.4 using the notice in the proof of Theorem 4.10. \square

Theorem 4.12 For any n , none of RT_2^{n+2} , $\text{RWKL}[\emptyset^{(n+1)}]$, $\text{FS}(n+2)$, $\text{TS}(n+2)$ and their stable versions admits a universal instance.

Proof. Fix an $n \in \omega$. Each of the above cited principles P satisfies $\text{STS}(n+2) \leq_c P$ and is a consequence of RT_2^{n+2} over ω -models. Cholak & al. [4] proved the existence of an ω -model of RT_2^{n+2} having only low_2 over $\emptyset^{(n)}$ sets. Apply Theorem 4.11. \square

We now turn to the proofs of Lemma 4.8, and Lemma 4.9.

Proof of Lemma 4.8. For each $e \in \omega$, let $Z_e = \{a \in \omega : f(e, a) = 1\}$. The proof is very similar to [20, Theorem 5.4.2.]. We build a \emptyset' -computable function $c : \omega \rightarrow \omega$ such that for all $e \in \omega$, if Z_e is infinite then it is not thin for c . Given such a function c , we can then apply Schoenfield's limit lemma to obtain a stable computable function $h : [\omega]^2 \rightarrow \omega$ such that for each $x \in \omega$, $\lim_s h(x, s) = c(x)$. Every set thin for h is thin for c , and therefore for all $e \in \omega$, if Z_e is infinite then it is not thin for h .

Suppose by Kleene's fixpoint theorem that we are given a Turing index d of the function c . The construction is done by a finite injury priority argument satisfying the following requirements for each $e, i \in \omega$:

$$\mathcal{R}_{e,i} : Z_e \text{ is finite or } (\exists a)[f(e, a) = 1 \text{ and } \Phi_d^{\emptyset'}(a) = i]$$

The requirements are ordered in a standard way, that is, following the pairing of the indexes. Notice that each of these requirement is Σ_2^f , and furthermore we can effectively find an index for each as such. Therefore, for each e and $i \in \omega$, we can effectively find an integer $m_{e,i}$ such that $\mathcal{R}_{e,i}$ is satisfied if and only if $m_{e,i} \in f''$. By Schoenfield's limit Lemma relativized to \emptyset' and low_2 -ness of f , there exists a \emptyset' -computable function $g : \omega^2 \rightarrow 2$ such that for all m , we have $m \in f'' \leftrightarrow \lim_s g(m, s) = 1$ and $m \notin f'' \leftrightarrow \lim_s g(m, s) = 0$. Notice that for all e and $i \in \omega$, $\mathcal{R}_{e,i}$ is satisfied if and only if $\lim_s g(m_{e,i}, s) = 1$.

At stage s , assume we have defined $c(u)$ for every $u < s$. If there exists a least strategy $\mathcal{R}_{e,i}$ (in priority order) with $\langle e, i \rangle < s$ such that $g(m_{e,i}, s) = 0$, set $c(s) = i$. Otherwise set $c(s) = 0$. This ends the construction. We now turn to the verification.

Claim. Every requirement $\mathcal{R}_{e,i}$ is satisfied.

Proof. By induction over ordered pairs $\langle e, i \rangle$ in lexicographic order. Suppose that $\mathcal{R}_{e',i'}$ is satisfied for all $\langle e', i' \rangle < \langle e, i \rangle$, but $\mathcal{R}_{e,i}$ is not satisfied. Then there exists a threshold $t \geq \langle e, i \rangle$ such that $g(m_{e',i'}, s) = 1$ for all $\langle e', i' \rangle < \langle e, i \rangle$ and $g(m_{e,i}, s) = 0$ whenever $s \geq t$. By construction, $c(s) = i$ for every $s \geq t$. As Z_e is infinite, there exists an element $s \in Z_e$ such that $c(s) = i$, so Z_e is not thin for c with witness i and therefore $\mathcal{R}_{e,i}$ is satisfied. Contradiction. \square

□

Proof of Lemma 4.9. For each $e \in \omega$, let $Z_e = \{a \in \omega : f(e, a) = 1\}$. The proof is very similar to [20, Theorem 5.4.2.]. We build a Δ_2^0 set U together with a stable computable linear order L such that U is the ω part of L , that is, U is the collection of elements L -below cofinitely many other elements. We furthermore ensure that for each $e \in \omega$, if Z_e is infinite, then it intersects both U and \overline{U} . Therefore, if Z_e is infinite, it is neither an ascending, nor a descending sequence in L as otherwise it would be included in either U or \overline{U} .

Assume by Kleene's fixpoint theorem that we are given the Turing index d of U . The set U is built by a finite injury priority construction with the following requirements for each $e \in \omega$:

- $\mathcal{R}_{2e} : Z_e \text{ is finite or } (\exists a)[f(e, a) = 1 \text{ and } \Phi_d^{\theta'}(a) = 1]$
- $\mathcal{R}_{2e+1} : Z_e \text{ is finite or } (\exists a)[f(e, a) = 1 \text{ and } \Phi_d^{\theta'}(a) = 0]$

Notice again that each of these requirement is Σ_2^f , and furthermore we can effectively find an index for each as such. Therefore, for each $i \in \omega$, we can effectively find an m_i such that R_i is satisfied if and only if $m_i \in f''$. By two applications of Schoenfield's limit Lemma and low_2 -ness of f , there exists a computable function $g : \omega^3 \rightarrow 2$ such that for all $m \in \omega$, we have $m \in f'' \leftrightarrow \lim_t \lim_s g(m, s, t) = 1$ and $m \notin f'' \leftrightarrow \lim_t \lim_s g(m, s, t) = 0$. Notice that for all $i \in \omega$,

$$R_i \text{ is satisfied} \leftrightarrow \lim_t \lim_s g(m_i, s, t) = 1$$

At stage 0, $U_0 = \emptyset$ and every integer is a *decision-maker* and *follows* itself. We say that \mathcal{R}_i *requires attention for* u at stage s if $i \leq u \leq s$, u is *decision-maker* and $g(m_i, s, u) = 0$. At stage $s + 1$, assume we have decided $u <_L v$ or $u >_L v$ for every $u, v < s$. Set $u <_L s$ if $u \in U_s$ and $u >_L s$ if $u \notin U_s$. Initially set $U_{s+1} = U_s$. For each decision-maker $u \leq s$ which has not been claimed at stage $s + 1$ and for which some requirement \mathcal{R}_i , $i < u$ requires attention, say that the least such \mathcal{R}_i *claims* u and act as follows.

- (a) If $i = 2e$ and $u \notin U_s$, then add $[u, s]$ to U_{s+1} . Elements of $[u + 1, s]$ *follow* u and are no more considered as decision-makers from now on and at any further stage.
- (b) If $i = 2e + 1$ and $u \in U_s$, then remove $[u, s]$ from U_{s+1} . As well, elements of $[u + 1, s]$ are no more decision-makers and *follow* u .

The go to next decision-maker $u \leq s$. This ends the construction. An immediate verification shows that at every stage,

- if u stops being a decision-maker it never becomes again a decision-maker
- if u follows v then $v \leq u$, v is a decision-maker, every w between v and u follows v and thus u will never follow any $w > v$.

So the decision-maker that u follows eventually stabilizes. As well, because g is limit-computable, each decision-maker eventually stops increasing the number of followers and therefore there are infinitely many decision-makers.

Claim. L is a linear order.

Proof. As L is a tournament, it suffices to check there is no 3-cycle. By symmetry, we check only the case where $u <_L s <_L v <_L u$ forms a 3-cycle with s the maximal element in $<_\omega$ order. By construction, this means that $u \in U_s$, $v \notin U_s$. If $u <_\omega v$, then $u \notin U_v$ and so there exists a decision-maker $w \leq_\omega u$ and an even number $i \leq w$ such

that \mathcal{R}_i requires attention for w at a stage $t \geq v$. Case (a) of the construction applies and the interval $[w + 1, t]$ is included U at least until stage s . As $v \in [w + 1, t]$, $v \in U_s$ contradicting our hypothesis. Case $u >_\omega v$ is symmetric. \square

Claim. U is Δ_2^0 .

Proof. Suppose for the sake of absurd that there exists a least element u entering U and leaving it infinitely many times. Such u must be a decision-maker, otherwise it would not be the least one. Let \mathcal{R}_i be the least requirement claiming u infinitely many times. As $\lim_s g(m_i, s, u)$ exists, it will claim u cofinitely many times and therefore u will be in U or in \overline{U} cofinitely many times. Contradiction. \square

It immediately follows that L is stable.

Claim. Every requirement \mathcal{R}_i is satisfied.

Proof. By induction over R_i in priority order. Suppose that R_j is satisfied for all $j < i$, but \mathcal{R}_i is not satisfied. Then there exists a threshold $t_0 \geq i$ such that $\lim_s g(m_j, s, t) = 1$ for all $j < i$ and $\lim_s g(m_i, s, t) = 0$ whenever $t \geq t_0$.

Then for every decision-maker $u \geq t_0$, \mathcal{R}_i will claim u cofinitely many times, and therefore u will be in U if i is even and in \overline{U} if i is odd. As every element follows the least decision-maker below itself, every v above the least decision-maker greater than t_0 will be in U if i is even and in \overline{U} if i is odd. So if Z_e is infinite, there will be such a $v \in Z_e$ satisfying \mathcal{R}_i . Contradiction. \square

\square

5. DEGREES BOUNDING THE ERDŐS MOSER THEOREM

Another approach to the strength analysis of Ramsey's theorem for pairs consists in seeing a coloring $f : [\omega]^2 \rightarrow 2$ as an infinite tournament T such that $T(x, y)$ holds for $x < y$ if and only if $f(x, y) = 1$. The Erdős Moser theorem states the existence of an infinite transitive subtournament, that is, an infinite subset on which the tournament behaves like a linear order. Therefore the Erdős Moser theorem can be seen as a principle reducing instances of RT_2^2 into instances of ADS.

Definition 5.1 (Erdős Moser theorem) A tournament T on a domain $D \subseteq \mathbb{N}$ is an ir-reflexive binary relation on D such that for all $x, y \in D$ with $x \neq y$, exactly one of $T(x, y)$ or $T(y, x)$ holds. A tournament T is *transitive* if the corresponding relation T is transitive in the usual sense. A tournament T is *stable* if $(\forall x \in D)[(\forall^\infty s)T(x, s) \vee (\forall^\infty s)T(s, x)]$. EM is the statement “Every infinite tournament T has an infinite transitive subtournament.” SEM is the restriction of EM to stable tournaments.

Bovykin and Weiermann proved in [2] that $EM + ADS$ is equivalent to RT_2^2 over RCA_0 , equivalence still holding between their stable versions. Lerman & al. [18] proved over $RCA_0 + B\Sigma_2^0$ that EM implies OPT and constructed an ω -model of EM not model of SRT_2^2 . Kreuzter & al. proved in [17] that SEM implies $B\Sigma_2^0$ over RCA_0 . Bienvenu & al. proved in [1] that $RCA_0 \vdash SEM \rightarrow RWKL$, hence there exists an ω -model of RRT_2^2 not model of SEM. Wang constructed in [28] an ω -model of $EM + COH$ not model of $STS(2)$. Finally, the author proved in [23] that $RCA_0 \vdash EM \rightarrow [STS(2) \vee COH]$.

The following notion of *minimal interval* plays a fundamental role in the analysis of EM. See [18] for a background analysis of EM.

Definition 5.2 (Minimal interval) Let T be an infinite tournament and $a, b \in T$ be such that $T(a, b)$ holds. The *interval* (a, b) is the set of all $x \in T$ such that $T(a, x)$ and $T(x, b)$ hold. Let $F \subseteq T$ be a finite transitive subtournament of T . For $a, b \in F$ such that $T(a, b)$ holds, we say that (a, b) is a *minimal interval of F* if there is no $c \in F \cap (a, b)$, i.e. no $c \in F$ such that $T(a, c)$ and $T(c, b)$ both hold.

We provide in the next subsections two different proofs of the existence of a low_2 degree bounding EM. More precisely, we construct a low_2 set G which is, up to finite changes, transitive for every infinite computable tournament.

The author proved in [23] that $[\text{STS}(2) \vee \text{COH}] \leq_c \text{EM}$. Therefore every low_2 degree bounding EM bounds also COH. The proof does not seem adaptable to prove that COH is a consequence of EM even in ω -models. However we can prove a weaker statement:

Lemma 5.3 For every set X , there exists an infinite X -computable tournament T such that for every infinite T -transitive subtournament U , $U \subseteq^* X$ or $U \subseteq^* \bar{X}$.

Proof. Fix a set X . We define a tournament T as follows: For each $a < b$, set $T(a, b)$ to hold iff $a \in X$ and $b \in X$ or $a \notin X$ and $b \notin X$. Suppose for the sake of absurd that U is an infinite transitive subtournament of T which intersects infinitely often X and \bar{X} . Take any $a, c \in U \cap X$ and $b, d \in U \cap \bar{X}$ such that $a < b < c < d$. Then $T(a, c)$, $T(c, b)$, $T(b, d)$ and $T(d, a)$ hold contradicting transitivity of U . \square

Using previous lemma, the constructed set G must be cohesive and therefore provides another proof of the existence of a low_2 cohesive set. Finally, we can deduce a statement slightly weaker than Theorem 4.10 simply by the existence of a low_2 degree bounding EM.

Lemma 5.4 There exists a set C such that there is no low_2 over C degree $\mathbf{d} \gg_{\text{SADS}} C$.

Proof. Fix a low_2 set $C \gg_{\text{EM}} \emptyset$ and a set X low_2 over C . By low_2 -ness of C , X is low_2 . Consider the stable coloring $f : [\omega]^2 \rightarrow 2$ constructed by Mileti in [20], such that X computes no infinite f -homogeneous set. We can see f as a stable tournament T such that for each $x < y$, $T(x, y)$ holds iff $f(x, y) = 1$. As $C \gg_{\text{EM}} \emptyset$, there exists an infinite C -computable transitive subtournament U of T . U is a stable linear order such that every infinite ascending or descending sequence is f -homogeneous. Therefore X computes no infinite ascending or descending sequence in U . \square

The following question remains open:

Question 5.5 Does EM admit a universal instance ?

5.1. A low_2 degree bounding EM using first jump control. The following theorem uses the proof techniques introduced in [4] for producing low_2 sets by controlling the first jump. It is done in the same spirit as Theorem 3.6 in [4].

Theorem 5.6 For every set $P \gg \emptyset'$, there exists a set $G \gg_{\text{EM}} \emptyset$ such that $G' \leq_T P$.

Before proving Theorem 5.6, we introduce the notion of *Erdős Moser condition*.

Definition 5.7 An *Erdős Moser condition* (EM condition) for an infinite tournament T is a Mathias condition (F, X) where

- (a) $F \cup \{x\}$ is T -transitive for each $x \in X$
- (b) X is included in a minimal T -interval of F .

Extension is usual Mathias extension. EM conditions have good properties for tournaments as state following lemmas. Given a tournament T and two sets E and F , we denote by $E \rightarrow_T F$ the formula $(\forall x \in E)(\forall y \in F)T(x, y)$ holds.

Lemma 5.8 Fix an EM condition (F, X) for a tournament T . For every $x \in F$, $\{x\} \rightarrow_T X$ or $X \rightarrow_T \{x\}$.

Proof. Fix an $x \in F$. Let (u, v) be the minimal T -interval containing X , where u, v may be respectively $-\infty$ and $+\infty$. By definition of interval, $\{u\} \rightarrow_T X \rightarrow_T \{v\}$. By definition of minimal interval, $T(x, u)$ or $T(v, x)$ holds. Suppose the former holds. By transitivity of $F \cup \{y\}$ for every $y \in X$, $T(x, y)$ holds, therefore $\{x\} \rightarrow_T Y$. In the latter case, by symmetry, $Y \rightarrow_T \{x\}$. \square

Lemma 5.9 Fix an EM condition $c = (F, X)$ for a tournament T , an infinite subset $Y \subseteq X$ and a finite T -transitive set $F_1 \subset X$ such that $F_1 < Y$ and $[F_1 \rightarrow_T Y \vee Y \rightarrow_T F_1]$. Then $d = (F \cup F_1, Y)$ is a valid extension of c .

Proof. Properties of a Mathias condition for d are immediate. We prove property (a). Fix an $x \in Y$. To prove that $F \cup F_1 \cup \{x\}$ is T -transitive, it suffices to check that there exists no 3-cycle in $F \cup F_1 \cup \{x\}$. Fix three elements $u < v < w \in F \cup F_1 \cup \{x\}$.

- Case 1: $\{u, v, w\} \cap F \neq \emptyset$. Then $u \in F$ as $F < F_1 < \{x\}$ and $u < v < w$. If $v \in F$ then using the fact that $F_1 \cup \{x\} \subset X$ and property (a) of condition c , $\{u, v, w\}$ is T -transitive. If $v \notin F$, then by Lemma 5.8, $\{u\} \rightarrow_T X (\supseteq F \cup \{x\})$ or $X \rightarrow_T \{u\}$ hence $\{u\} \rightarrow_T \{v, w\}$ or $\{v, w\} \rightarrow_T \{u\}$ so $\{u, v, w\}$ is T -transitive.
- Case 2: $\{u, v, w\} \cap F = \emptyset$. Then at least $u, v \in F_1$ because $F_1 < \{x\}$. If $w \in F_1$, then $\{u, v, w\}$ is T -transitive by T -transitivity of F_1 . Otherwise, as $F_1 \rightarrow_T Y$ or $Y \rightarrow_T F_1$, $\{u, v\} \rightarrow_T \{w\}$ or $\{w\} \rightarrow_T \{u, v\}$ and $\{u, v, w\}$ is T -transitive.

We now prove property (b). Let (u, v) be the minimal T -interval of F in which X (hence Y) is included by property (b) of condition c . u and v may be respectively $-\infty$ and $+\infty$. By assumption, either $F_1 \rightarrow_T Y$ or $Y \rightarrow_T F_1$. As F_1 is a finite T -transitive set, it has a minimal and a maximal element, say x and y . If $F_1 \rightarrow_T Y$ then Y is included in the T -interval (y, v) . Symmetrically, if $Y \rightarrow_T F_1$ then Y is included in the T -interval (u, x) . To prove minimality for the first case, assume that some w is in the interval (y, v) . Then $w \notin F$ by minimality of the interval (u, v) w.r.t. F , and $w \notin F_1$ by maximality of y . Minimality for the second case holds by symmetry. \square

Proof of Theorem 5.6. Let C be a low set such that there exists a uniformly C -computable enumeration \vec{T} of infinite tournaments containing every computable tournament. Note that $P \gg C'$. Our forcing conditions are tuples (σ, F, X) where $\sigma \in \omega^{<\omega}$ and the following holds:

- (a) (F, X) forms a Mathias condition and X is a set low over C .
- (b) $(F \setminus [0, \sigma(v)], X)$ is an EM condition for T_v for each $v < |\sigma|$.

A condition $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ extends a condition (σ, F, X) if $\sigma \preceq \tilde{\sigma}$ and (\tilde{F}, \tilde{X}) Mathias extends (F, X) . A set G satisfies the condition (σ, F, X) if $G \setminus [0, \sigma(v)]$ is T_v -transitive for each

$v < |\sigma|$ and G satisfies the Mathias condition (F, X) . An *index* of a condition (σ, F, X) is a code of the tuple $\langle \sigma, F, e \rangle$ where e is a lowness index of X .

The first lemma simply states that we can ensure that G will be infinite and eventually transitive for each tournament in \vec{T} .

Lemma 5.10 For every condition $c = (\sigma, F, X)$ and every $i, j \in \omega$, one can P -compute an extension $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ such that $|\tilde{\sigma}| \geq i$ and $|\tilde{F}| \geq j$ uniformly from i, j and an index of c .

Proof. Let x be the first element of X . As X is low over C , x can be found C' -computably from a lowness index of X . The condition $(\tilde{\sigma}, F, X)$ is a valid extension of c where $\tilde{\sigma} = \sigma \cap x \dots x$ so that $|\tilde{\sigma}| \geq i$. It suffices to prove that we can C' -compute an extension $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ with $|\tilde{F}| > |F|$ and iterate the process. Define the computable coloring $g : X \rightarrow 2^{|\tilde{\sigma}|}$ by $g(s) = \rho$ where $\rho \in 2^{|\tilde{\sigma}|}$ such that $\rho(v) = 1$ iff $T_v(x, s)$ holds. One can find uniformly in P a $\rho \in 2^{|\tilde{\sigma}|}$ such that the following C -computable set is infinite:

$$Y = \{s \in X \setminus \{x\} : g(s) = \rho\}$$

By Lemma 5.9, $((F \cup \{x\}) \setminus [0, \tilde{\sigma}(v)], Y)$ is a valid EM extension for T_v . As Y is low over C , $(\tilde{\sigma}, F \cup \{x\}, Y)$ is a valid extension for c . \square

It remains to be able to decide $e \in (G \oplus C)'$ uniformly in e . We first need to define a forcing relation.

Definition 5.11 Fix a condition $c = (\sigma, F, X)$ and two integers e and x .

1. $c \Vdash \Phi_e^{G \oplus C}(x) \uparrow$ if $\Phi_e^{(F \cup F_1) \oplus C}(x) \uparrow$ for all finite subsets $F_1 \subseteq X$ such that F_1 is T_v -transitive simultaneously for each $v < |\sigma|$.
2. $c \Vdash \Phi_e^{G \oplus C}(x) \downarrow$ if $\Phi_e^{F \oplus C}(x) \downarrow$.

Note that the way we defined our forcing relation $c \Vdash \Psi_e^{G \oplus C}(x) \uparrow$ differs slightly from the “true” forcing notion \Vdash^* inherited by the notion of satisfaction of G . The true forcing definition of this statement is the following:

$c \Vdash^* \Phi_e^{G \oplus C}(x) \uparrow$ if $\Phi_e^{(F \cup F_1) \oplus C}(x) \uparrow$ for all finite *extensible* subsets $F_1 \subseteq X$ such that F_1 is T_v -transitive simultaneously for each $v < |\sigma|$, i.e. for all finite subsets $F_1 \subseteq X$ such that there exists an extension $d = (\tilde{\sigma}, F \cup F_1, \tilde{X})$.

However $c \Vdash^* \Phi_e^{G \oplus C}(x) \uparrow$ is not a Π_1^0 statement whereas $c \Vdash \Phi_e^{G \oplus C}(x) \uparrow$ is. In particular the fact that $c \not\Vdash \Phi_e^{G \oplus C}(x) \uparrow$ does not mean that c has an extension forcing its negation. This subtlety is particularly important in Lemma 5.13. The following lemma gives a sufficient constraint, namely being included in a part of a particular partition, on finite transitive sets to ensure that they are *extensible*.

Lemma 5.12 Let $c = (\sigma, F, X)$ be a condition and $E \subseteq X$ be a finite set. There exists a $2^{|\sigma|}$ partition $(E_\rho : \rho \in 2^{|\sigma|})$ of E and an infinite set $Y \subseteq X$ low over C such that $E < Y$ and for all $\rho \in 2^{|\sigma|}$ and $v < |\sigma|$, if $\rho(v) = 0$ then $E_\rho \rightarrow_{T_v} Y$ and if $\rho(v) = 1$ then $Y \rightarrow_{T_v} E_\rho$.

Moreover this partition and a lowness index of Y can be uniformly P -computed from an index of c and the set E .

Proof. Given a set E , define P_E to be the finite set of ordered $2^{|\sigma|}$ -partitions of E , that is,

$$P_E = \{ \langle E_\rho : \rho \in 2^{|\sigma|} \rangle : \bigcup_{\rho \in 2^{|\sigma|}} E_\rho = E \text{ and } \rho \neq \xi \rightarrow E_\rho \cap E_\xi = \emptyset \}$$

Define the C -computable coloring $g : X \rightarrow P_E$ by $g(x) = (E_\rho^x : \rho \in 2^{|\sigma|})$ where $E_\rho^x = \{a \in E : (\forall v < |\sigma|)[T_v(a, x) \text{ holds iff } \rho(v) = 0]\}$. One can find uniformly in P a partition $(E_\rho : \rho \in 2^{|\sigma|})$ such that the following C -computable set is infinite:

$$Y = \{x \in X \setminus E : g(x) = (E_\rho : \rho \in 2^{|\sigma|})\}$$

By definition of g , for all $\rho \in 2^{|\sigma|}$ and $v < |\sigma|$, if $\rho(v) = 0$ then $E_\rho \rightarrow_{T_v} Y$ and if $\rho(v) = 1$ then $Y \rightarrow_{T_v} E_\rho$. \square

We are now ready to prove the key lemma of this forcing, stating that we can P -decide whether or not $e \in G'$ for any $e \in \omega$.

Lemma 5.13 For every condition (σ, F, X) and every $e \in \omega$, there exists an extension $d = (\tilde{\sigma}, \tilde{F}, \tilde{X})$ such that one of the following holds:

1. $d \Vdash \Phi_e^{G \oplus C}(e) \downarrow$
2. $d \Vdash \Phi_e^{G \oplus C}(e) \uparrow$

This extension can be P -computed uniformly from an index of c and e . Moreover there is a C' -computable procedure to decide which case holds from an index of d .

Proof. Let $k = |\sigma|$. Using a C' -computable procedure, we can decide from an index of c and e whether there exists a finite set $E \subset X$ such that for every 2^k -partition $(E_i : i < 2^k)$ of E , there exists an $i < 2^k$ and a subset $F_1 \subseteq E_i$ T_v -transitive simultaneously for each $v < k$ and satisfying $\Phi_e^{(F \cup F_1) \oplus C}(e) \downarrow$.

1. If such a set E exists, it can be C' -computably found. By Lemma 5.12, one can P -computably find a 2^k -partition $(E_\rho : \rho \in 2^k)$ of E and a set $Y \subseteq X$ low over C such that for all $\rho \in 2^k$ and $v < k$, if $\rho(v) = 0$ then $E_\rho \rightarrow_{T_v} Y$ and if $\rho(v) = 1$ then $Y \rightarrow_{T_v} E_\rho$. We can C' -computably find a $\rho \in 2^k$ and a set $F_1 \subseteq E_\rho$ which is T_v -transitive simultaneously for each $v < k$ and satisfying $\Phi_e^{(F \cup F_1) \oplus C}(e) \downarrow$. By Lemma 5.9, $(F \setminus [0, \sigma(v)] \cup F_1, Y)$ is a valid EM extension of $(F \setminus [0, \sigma(v)], X)$ for T_v for each $v < k$. As Y is low over C , $(\sigma, F \cup F_1, Y)$ is a valid extension of c forcing $\Phi_e^{G \oplus C}(e) \downarrow$.
2. If no such set exists, then by compactness, the $\Pi_1^{0,C}$ class of all 2^k -partitions $(X_i : i < 2^k)$ of X such that for every $i < 2^k$ and every finite set $F_1 \subseteq X_i$ which is T_v -transitive simultaneously for each $v < k$, $\Phi_e^{(F \cup F_1) \oplus C}(e) \uparrow$ is non-empty. In other words, the $\Pi_1^{0,C}$ class of all 2^k -partitions $(X_i : i < 2^k)$ of X such that for every $i < 2^k$, $(\sigma, F, X_i) \Vdash \Phi_e^{G \oplus C}(e) \uparrow$ is non-empty. By the relativized low basis theorem, there exists a 2^k -partition $(X_i : i < 2^k)$ of X low over C . Furthermore, a lowness index for this partition can be uniformly C' -computably found. Using P , one can find an $i < 2^k$ such that X_i is infinite. (σ, F, X_i) is a valid extension of c forcing $\Phi_e^{G \oplus C}(e) \uparrow$. \square

Using Lemma 5.10 and Lemma 5.13, one can P -compute an infinite decreasing sequence of conditions $c_0 = (\epsilon, \emptyset, \omega) \geq c_1 \geq \dots$ such that for each $s > 0$

1. $|\sigma_s| \geq s, |F_s| \geq s$
2. $c_s \Vdash \Phi_s^{G \oplus C}(s) \downarrow$ or $c_s \Vdash \Phi_s^{G \oplus C}(s) \uparrow$

where $c_s = (\sigma_s, F_s, X_s)$. The resulting set $G = \bigcup_s F_s$ is T_v -transitive up to finite changes for each $v \in \omega$ and $G' \leq_T P$. \square

5.2. A low₂ degree bounding EM using second jump control. We now use the second proof technique used in [4] for producing a low₂ set. It consists of directly controlling the second jump of the produced set.

Theorem 5.14 There exists a low₂ degree bounding EM.

Proof. Similar to Theorem 5.6, we fix a low set C such that there exists a uniformly C -computable enumeration \vec{T} of infinite tournaments containing every computable tournament. In particular $P \gg C'$.

Our forcing conditions are the same as in Theorem 5.6. We can release the constraints of infinity and lowness over C for X in a condition (σ, F, X) . This gives the notion of *precondition*. The forcing relations extend naturally to preconditions.

Definition 5.15 Fix a finite set of Turing indexes \vec{e} . A condition (σ, F, X) is \vec{e} -small if there exists a number x and a sequence $(\sigma_i, F_i, X_i : i < n)$ such that for each $i < n$

- (i) (σ_i, F_i, X_i) is a precondition extending c
- (ii) $(X_i : i < n)$ is a partition of $X \cap (x, +\infty)$
- (iii) $\max(X_i) < x$ or $(\sigma_i, F \cup F_i, X_i) \Vdash (\exists e \in \vec{e})(\exists y < x)\Phi_e^{G \oplus C}(y) \uparrow$

A condition is \vec{e} -large if it is not \vec{e} -small.

A condition $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ is a *finite extension* of (σ, F, X) if $\tilde{X} =^* X$. Finite extensions do not play the same fundamental role as in the original forcing in [4] as adding elements to the set F may require to remove infinitely many elements of the promise set X to obtain a valid extension. We nevertheless prove the following traditional lemma.

Lemma 5.16 Fix an \vec{e} -large condition $c = (\sigma, F, X)$.

1. If $\vec{e}' \subseteq \vec{e}$ then c is \vec{e}' -large.
2. If d is a finite extension of c then d is \vec{e} -large.

Proof. Clause 1 is trivial as \vec{e} appears only in a universal quantification in the definition of \vec{e} -largeness. We prove clause 2. Let $d = (\tilde{\sigma}, \tilde{F}, \tilde{X})$ be an \vec{e} -small finite extension of c . We will prove that c is \vec{e} -small. Let $x \in \omega$ and $(\sigma_i, F_i, X_i : i < n)$ witness \vec{e} -smallness of d . Let $y = \max(x, X \setminus \tilde{X})$. For each $i < n$, set $\tilde{F}_i = (\tilde{F} \setminus F) \cup F_i$ and $\tilde{X}_i = X_i \cap (y, +\infty)$. Then y and $(\sigma_i, \tilde{F}_i, \tilde{X}_i : i < n)$ witness \vec{e} -smallness of c . \square

Lemma 5.17 There exists a C'' -effective procedure to decide, given an index of a condition c and a finite set of Turing indexes \vec{e} , whether c is \vec{e} -large. Furthermore, if c is \vec{e} -small, there exists sets $(X_i : i < n)$ low over C witnessing this, and one may C' -compute a value of n , x , lowness indexes for $(X_i : i < n)$ and the corresponding sequences $(\sigma_i, F_i, X_i : i < n)$ which witness that c is \vec{e} -small.

Proof. Fix a condition $c = (\sigma, F, X)$ The predicate “ (σ, F, X) is \vec{e} -small” can be expressed as a Σ_2^0 statement

$$(\exists z)(\exists Z)P(z, Z, F, X, \vec{v}, \vec{e})$$

where P is a $\Pi_1^{0,C}$ predicate. Here z codes n and x , and Z codes $(X_i : i < n)$. $(\exists Z)P(z, Z, F, X, \sigma, \vec{e})$ is a $\Pi_1^{0,C \oplus X}$ predicate by compactness. As X is low over C and F and σ are finite, one can compute a $\Delta_2^{0,C}$ index for the same predicate P with parameter z , an index of c and \vec{e} , from a lowness index for X , F and σ . Therefore there exists a $\Sigma_2^{0,C}$ statement with parameters an index of c and \vec{e} which holds iff c is \vec{e} -small.

If c is \vec{e} -small, there exists sets $(X_i : i < n)$ low over X (hence low over C) witnessing it by the low basis theorem relativized to C . By the uniformity of the proof of the low basis theorem, one can compute lowness indexes of $(X_i : i < n)$ uniformly from a lowness index of X . \square

As the extension produced in Lemma 5.10 is not a finite extension, we need to refine it to ensure largeness preservation.

Lemma 5.18 For every \vec{e} -large condition $c = (\sigma, F, X)$ and every $i, j \in \omega$, one can P -compute an \vec{e} -large extension $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ such that $\tilde{\sigma} \geq i$ and $|\tilde{F}| \geq j$ uniformly from an index of c , i , j and \vec{e} .

Proof. Let x be the first element of X . As X is low over C , x can be found C' -computably from a lowness index of X . The condition $d = (\tilde{\sigma}, F, X)$ is a valid extension of c where $\tilde{\sigma} = \sigma \cap x \dots x$ so that $|\tilde{\sigma}| \geq i$. As d is a finite extension of c , it is \vec{e} -large by Lemma 5.16. It suffices to prove that we can C' -compute an \vec{e} -large extension $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ with $|\tilde{F}| > |F|$ and iterate the process. Define the C -computable coloring $g : X \rightarrow 2^{|\tilde{\sigma}|}$ as in Lemma 5.10. For each $\rho \in 2^{|\tilde{\sigma}|}$, define the following set:

$$Y_\rho = \{s \in X \setminus \{x\} : g(s) = \rho\}$$

There must be a $\rho \in 2^{|\tilde{\sigma}|}$ such that Y_ρ is infinite and $(\tilde{\sigma}, F \cup \{x\}, Y_\rho)$ is \vec{e} -large, otherwise the witnesses of \vec{e} -smallness for each $\rho \in 2^{|\tilde{\sigma}|}$ would witness \vec{e} -smallness of c . By Lemma 5.17, one can C'' -find a $\rho \in 2^{|\tilde{\sigma}|}$ such that $(\tilde{\sigma}, F \cup \{x\}, Y_\rho)$ is \vec{e} -large. As seen in Lemma 5.18, $(\tilde{\sigma}, F, \{x\}, Y_\rho)$ is a valid extension. \square

The following lemma is a refinement of Lemma 5.12 controlling largeness preservation.

Lemma 5.19 Let $c = (\sigma, F, X)$ be an \vec{e} -large condition and $E \subseteq X$ be a finite set. There is a $2^{|\sigma|}$ partition $(E_\rho : \rho \in 2^{|\sigma|})$ of E and an infinite set $Y \subseteq X$ low over C such that $E < Y$ and

1. for all $\rho \in 2^{|\sigma|}$ and $v < |\sigma|$, if $\rho(v) = 0$ then $E_\rho \rightarrow_{T_v} Y$ and if $\rho(v) = 1$ then $Y \rightarrow_{T_v} E_\rho$.
2. $(\sigma, F \cup F_1, Y)$ is an \vec{e} -large condition extending d for every $\rho \in 2^{|\sigma|}$ and every finite set $F_1 \subseteq E_\rho$ which is T_v -transitive for each $v < |\sigma|$

Moreover this partition and a lowness index of Y can be uniformly C'' -computed from an index of c and the set E .

Proof. Given a set E , recall from Lemma 5.12 that P_E is the finite set or ordered 2^k -partitions of E . Define again the computable coloring $g : X \rightarrow P_E$ by $g(x) = \langle E_\rho^x : \rho \in 2^{|\sigma|} \rangle$ where $E_\rho^x = \{a \in E : (\forall v < |\sigma|)[T_v(a, x) \text{ holds iff } \rho(v) = 0]\}$. If for each partition $(E_\rho : \rho \in 2^{|\sigma|})$, there exists a $\rho \in 2^{|\sigma|}$ and a $F_1 \subseteq E_\rho$ which is T_v -transitive simultaneously for each $v < |\sigma|$ and such that $(\sigma, F \cup F_1, Y)$ is \vec{e} -small

where

$$Y = \{x \in X \setminus E : g(x) = (E_\rho : \rho \in 2^{|\sigma|})\}$$

Then we could construct a witness of \vec{e} -smallness of c using smallness witnesses of $(\sigma, F \cup F_1, Y)$ for each partition $(E_\rho : \rho \in 2^{|\sigma|})$. Therefore there must exist a partition $(E_\rho : \rho \in 2^{|\sigma|})$ such that Y is infinite and $d = (\sigma, F \cup F_1, Y)$ is \vec{e} -large for every $\rho \in 2^{|\sigma|}$ and every $F_1 \subseteq E_\rho$ which is T_v -transitive for each $v < |\sigma|$.

By Lemma 5.17, such partition can be found C'' -computably. By definition of g , for all $\rho \in 2^{|\sigma|}$ and $v < k$, if $\rho(v) = 0$ then $E_\rho \rightarrow_{T_v} Y$ and if $\rho(v) = 1$ then $Y \rightarrow_{T_v} E_\rho$. Therefore, by Lemma 5.9, $((F \setminus [0, \sigma(v)]) \cup F_1, Y)$ is a valid EM extension of $(F \setminus [0, \sigma(v)], X)$ for T_v for each $v < |\sigma|$, so d is a valid condition. \square

Lemma 5.20 Suppose that $c = (\sigma, F, X)$ is \vec{e} -large. For every $y \in \omega$ and $e \in \vec{e}$, there exists an \vec{e} -large extension d such that $d \Vdash \Phi_e^{G \oplus C}(y) \downarrow$. Furthermore, an index for d can be computed from an oracle for C' from an index of c , e and y .

Proof. Let $k = |\sigma|$. As c is \vec{e} -large, then by a compactness argument, there exists a finite set $E \subset X$ such that for every 2^k -partition $(E_i : i < 2^k)$ of E , there exists an $i < k$ and a finite subset $F_1 \subseteq E_i$ which is T_v -transitive simultaneously for each $v < k$, and $\Phi_e^{(F \cup F_1) \oplus C}(y) \downarrow$. Moreover this set E can be C' -computably found. By Lemma 5.19, on can uniformly C'' -find a partition $(E_\rho : \rho \in 2^k)$ of E and a lowness index for an infinite set $Y \subseteq X$ low over C such that

1. for all $\rho \in 2^k$ and $v < k$, if $\rho(v) = 0$ then $E_\rho \rightarrow_{T_v} Y$ and if $\rho(v) = 1$ then $Y \rightarrow_{T_v} E_\rho$.
2. $(\sigma, F \cup F_1, Y)$ is an \vec{e} -large condition extending c for every $\rho \in 2^k$ and every finite set $F_1 \subseteq E_\rho$ which is T_v -transitive for each $v < k$

We can then produce by a C' -computable search a $\rho \in 2^k$ and a finite set $F_1 \subseteq E_\rho$ which is T_v -transitive for each $v < k$ and such that $\Phi_e^{(F \cup F_1) \oplus C}(y) \downarrow$. By Lemma 5.9, $((F \setminus [0, \sigma(v)]) \cup F_1, Y)$ is a valid EM extension of $(F \setminus [0, \sigma(v)], X)$ for T_v for each $v < k$. As Y is low over C , $(\sigma, F \cup F_1, Y)$ is a valid \vec{e} -large extension. \square

Lemma 5.21 Suppose that $c = (\sigma, F, X)$ is \vec{e} -large and $(\vec{e} \cup \{u\})$ -small. There exists a \vec{e} -large extension d such that $d \Vdash \Phi_u^{G \oplus C}(y) \uparrow$ for some $y \in \omega$. Furthermore one can find an index for d by applying a C'' -computable function to an index of c , \vec{e} and u .

Proof. By Lemma 5.17, we may choose the sets $(X_i : i < n)$ witnessing that c is $(\vec{e} \cup \{u\})$ -small to be low over C . Fix the corresponding x and $(\sigma_i, F_i : i < n)$. Consider the i 's such that $(\sigma_i, F_i, X_i) \Vdash \Phi_u^{G \oplus C}(y) \uparrow$ for some $y < x$. As c is \vec{e} -large, there must be one such $i < n$ such that (σ_i, F_i, X_i) is an \vec{e} -large condition. By Lemma 5.17 we can find C'' -computably such $i < n$. (σ_i, F_i, X_i) is the desired extension. \square

Using previous lemmas, we can C'' -compute an infinite descending sequence of conditions $c_0 = (\epsilon, \emptyset, \omega) \geq c_1 \geq \dots$ together with an infinite increasing sequence of Turing indexes $\vec{e}_0 = \emptyset \subseteq \vec{e}_1 \subseteq \dots$ such that for each $s > 0$

1. $|\sigma_s| \geq s$, $|F_s| \geq s$, c_s is \vec{e}_s -large
2. Either $s \in \vec{e}_s$ or $c_s \Vdash \Phi_s^{G \oplus C}(y) \uparrow$ for some $y \in \omega$
3. $c_s \Vdash \Phi_e^{G \oplus C}(x) \downarrow$ if $s = \langle e, x \rangle$ and $e \in \vec{e}_s$

where $c_s = (\sigma_s, F_s, X_s)$. The resulting set $G = \bigcup_s F_s$ is T_v -transitive up to finite changes simultaneously for each $v \in \omega$ and $G'' \leq_T C'' \leq_T \emptyset''$. \square

6. DEGREE BOUNDING THE RAINBOW RAMSEY THEOREM

The rainbow Ramsey theorem intuitively states that when a coloring over tuples uses each color a bounded number of times then it has an infinite subset on which each color is used at most once. This statement has been extensively studied over the past few years [8, 7, 26, 23]. Remarkably, the restriction of the rainbow Ramsey theorem to coloring over pairs of integers coincides with a well-known notion of algorithmic randomness.

Definition 6.1 (Rainbow Ramsey theorem) Let $n, k \in \omega$. A coloring function $f : [\omega]^n \rightarrow \omega$ is k -bounded if for every $y \in \omega$, $|f^{-1}(y)| \leq k$. A set R is a *rainbow* for f if $f \upharpoonright [R]^n$ is injective. RRT_k^n is the statement “Every k -bounded function $f : [\omega]^n \rightarrow \omega$ has an infinite rainbow”.

A proof of the rainbow Ramsey theorem is due to Galvin who noticed that it follows easily from Ramsey’s theorem. Hence every computable 2-bounded coloring function f over n -tuples has an infinite Π_n^0 rainbow. Csisma and Mileti proved in [8] that every 2-random is RRT_2^2 -bounding and deduced that RRT_2^2 implies neither SADS nor WKL_0 over ω -models. Conidis & Slaman adapted in [7] the argument from Cisma and Mileti to obtain $\text{RCA}_0 \vdash 2\text{-RAN} \rightarrow \text{RRT}_2^2$.

Definition 6.2 A function $f : \omega \rightarrow \omega$ is *diagonally non-computable (DNC) relative to X* if $f(e) \neq \Phi_e^X(e)$ for each $e \in \omega$. $\text{DNR}[\emptyset']$ is the statement “For every set X , there exists a function DNC relative to the jump of X ”.

Theorem 6.3 (J.S. Miller [21]) RRT_2^2 and $\text{DNR}[\emptyset']$ are computably equivalent.

Corollary 6.4 RRT_2^2 admits a universal instance.

Proof. If P and Q are two principles computably equivalent and Q admits a universal instance, then so does P . As $\text{DNR}[\emptyset']$ admits a universal instance (any function DNC relative to \emptyset'), so does RRT_2^2 . \square

Corollary 6.5 For every $X \gg \emptyset'$, there exists a $Y \gg_{\text{RRT}_2^2} \emptyset$ such that $Y' \leq_T X$.

Proof. Let $f : [\omega]^2 \rightarrow \omega$ be a universal instance of RRT_2^2 . By Csisma & Mileti [8], $\text{RRT}_2^2 \leq_c \text{RT}_2^2$, so there exists a computable coloring $g : [\omega]^2 \rightarrow 2$ such that every infinite g -homogeneous set computes an infinite f -rainbow, hence bounds RRT_2^2 . By Cholak & al. [4], for every $X \gg \emptyset'$ there exists an infinite f -homogeneous set H such that $H' \leq_T X$. In particular $H \gg_{\text{RRT}_2^2} \emptyset$. \square

Corollary 6.6 There exists a low₂ degree bounding RRT_2^2 .

Proof. By the relativized low basis theorem, there exists a set $X \gg \emptyset'$ low over \emptyset' . By Corollary 6.5, there exists a set $Y \gg_{\text{RRT}_2^2} \emptyset$ such that $Y' \leq_T X$, hence $Y'' \leq_T X' \leq_T \emptyset''$. So Y is low₂. \square

We can generalize Corollary 6.6 to colorings over arbitrary tuples. For this, we need to restrict ourselves to the study of a particular class of colorings.

Definition 6.7 A coloring $f : [\omega]^{n+1} \rightarrow \omega$ is *normal* if $f(\sigma, a) \neq f(\tau, b)$ for each $\sigma, \tau \in [\omega]^n$, whenever $a \neq b$.

Wang proved in [26] that for every 2-bounded coloring $f : [\omega]^n \rightarrow \omega$, every f -random computes an infinite set X on which f is normal. The author refined in [23] this result by proving that every function d.n.c. relative to f computes such a set.

Theorem 6.8 For each $n \geq 0$, there exists a set $X \gg_{\text{RRT}_2^{n+2}} \emptyset \text{ low}_2$ over $\emptyset^{(n)}$.

Proof. We prove by induction over n that for every set A there exists a set $X \text{ low}_2$ over $A^{(n)}$ such that $X \gg_{\text{RRT}_2^{n+2}} A$. Case $n = 0$ is a relativization of Corollary 6.6. Suppose for each set A , there exists a set $X \text{ low}_2$ over $A^{(n)}$ such that $X \gg_{\text{RRT}_2^{n+2}} A$. Fix a set A , an A -random set $R \text{ low}_2$ over A and a set $C \text{ low}_2$ over $A \oplus R$ such that $C' \gg (A \oplus R)'$. In particular $R \oplus C$ is low_2 over A . By induction hypothesis, there exists a set $X \text{ low}_2$ over $(A \oplus R \oplus C)^{(n+1)}$ such that $X \gg_{\text{RRT}_2^{n+2}} (A \oplus R \oplus C)'$. In particular X is low_2 over $A^{(n+1)}$. We claim that $X \gg_{\text{RRT}_2^{n+3}} A$.

Fix an A -computable 2-bounded coloring $f : [\omega]^{n+3} \rightarrow \omega$. By relativizing Lemma 4.3 in [26], every A -random computes an infinite set Y such that f restricted to Y is normal. So $X \oplus R$ computes such a set Y . For each $\sigma, \tau \in [Y]^{n+2}$, let

$$U_{\sigma, \tau} = \{s \in Y : f(\sigma, s) = f(\tau, s)\}$$

By Jockusch & Frank [16], as $C' \gg (A \oplus R)'$, $A \oplus R \oplus C$ computes an infinite \vec{U} -cohesive set $Z \subseteq Y$. In particular the following limit exists

$$\tilde{f}(\sigma) = \lim_{s \in Z} \min\{\tau \leq_{\text{lex}} \sigma : f(\sigma, s) = f(\tau, s)\}$$

\tilde{f} is a 2-bounded $(A \oplus R \oplus C)'$ -computable coloring of $(n+2)$ -tuples, so X bounds an infinite \tilde{f} -rainbow H . $A \oplus H$ computes an infinite f -rainbow, so X bounds an infinite f -rainbow. \square

6.1. A stable rainbow Ramsey theorem. A common process in the strength analysis of a principle consists of splitting the statement into a stable and a cohesive version. The standard notion of stability does not apply for the rainbow Ramsey theorem as no stable coloring is k -bounded for some $k \in \omega$. Nevertheless one can define certain notions of stability for the rainbow Ramsey theorem [23]. Mileti proved in [20] that the only Δ_2^0 degree bounding SRT_2^2 is \emptyset' . In fact, his priority argument can be adapted to prove the same result on a much weaker principle coinciding with a stable version of the rainbow Ramsey theorem for pairs.

Definition 6.9 A coloring $f : [\omega]^2 \rightarrow \omega$ is *rainbow-stable* if for every $x \in \omega$, one of the following holds:

- (a) There exists a $y \neq x$ such that $(\forall^\infty s) f(x, s) = f(y, s)$
- (b) $(\forall^\infty s) |\{y \neq x : f(x, s) = f(y, s)\}| = 0$

SRRT_2^2 is the statement “every rainbow-stable 2-bounded coloring $f : [\omega]^2 \rightarrow \omega$ has a rainbow.”

Introduced by the author in [23], he proved that SRRT_2^2 is computably reducible to SEM and STS(2). This principle admits many computably equivalent formulations.

We are particularly interested in a characterization which can be seen as a stable notion of $\text{DNR}[\emptyset']$.

Definition 6.10 Given a function $f : \omega \rightarrow \omega$, a function g is f -diagonalizing if $(\forall x)[f(x) \neq g(x)]$. $\text{SDNR}[\emptyset']$ is the statement “Every Δ_2^0 function $f : \omega \rightarrow \omega$ has an f -diagonalizing function”.

Theorem 6.11 (Patey [23]) SRRT_2^2 and $\text{SDNR}[\emptyset']$ are computably equivalent.

The following theorem extends Milet’s result to $\text{SDNR}[\emptyset']$. As $\text{SDNR}[\emptyset']$ is computably below many stable principles, we shall deduce a few more non-universality results.

Theorem 6.12 For every Δ_2^0 incomplete set X , there exists a Δ_2^0 function $f : \omega \rightarrow \omega$ such that X computes no f -diagonalizing function.

Corollary 6.13 A Δ_2^0 degree \mathbf{d} bounds SRRT_2^2 iff $\mathbf{d} = \mathbf{0}'$.

Proof. As $\text{SRRT}_2^2 \leq_c \text{SRT}_2^2$, any computable instance of SRRT_2^2 has a Δ_2^0 solution. So $\mathbf{0}'$ bounds SRRT_2^2 . If \mathbf{d} is incomplete, then by Theorem 6.12 and by $\text{SRRT}_2^2 =_c \text{SDNR}[\emptyset']$, there is a computable instance of SRRT_2^2 such that \mathbf{d} bounds no solution. \square

Corollary 6.14 No statement P such that $\text{SRRT}_2^2 \leq_c P \leq_c \text{SRT}_2^2$ admits a universal instance.

Proof. By [13, Corollary 4.6] every Δ_2^0 set or its complement has an incomplete Δ_2^0 infinite subset. As $P \leq_c \text{SRT}_2^2 \leq_c \text{D}_2^2$, every computable instance U of P has a Δ_2^0 incomplete solution X . By Theorem 6.12, there exists a computable coloring $f : [\omega]^2 \rightarrow \omega$ such that X computes no infinite f -rainbow. As $\text{SRRT}_2^2 \leq_c P$, there exists a computable instance of P such that X does not compute a solution to it. Hence U is not a universal instance of P . \square

Corollary 6.15 None of SRRT_2^2 , SEM , $\text{STS}(2)$ and $\text{SFS}(2)$ admits a universal instance.

Proof of Theorem 6.12. The proof is an adaptation of [20, Theorem 5.3.7]. Suppose that D is a Δ_2^0 incomplete set. We will construct a Δ_2^0 coloring $f : \omega \rightarrow \omega$ such that D does not compute any f -diagonalizing function. We want to satisfy the following requirements for each $e \in \omega$:

\mathcal{R}_e : If Φ_e^D is total, then there is an a such that $\Phi_e^D(a) = f(a)$.

For each $e \in \omega$, define the partial function u_e by letting $u_e(a)$ be the use of Φ_e^D on input a if $\Phi_e^D(a) \downarrow$ and letting $u_e(a) \uparrow$ otherwise. We can assume w.l.o.g. that whenever $u_e(a) \downarrow$ then $u_e(a) \geq a$. Also define a computable partial function θ by letting $\theta(a) = (\mu t)[a \in \emptyset'_t]$ if $a \in \emptyset'$ and $\theta(a) \uparrow$ otherwise.

The local strategy for satisfying a single requirement \mathcal{R}_e works as follows. If \mathcal{R}_e receives attention at stage s , this strategy does the following. First it identifies a number $a \geq e$ that is not restrained by strategies of higher priority such that the following conditions holds:

- (i) $\Phi_{e,s}^{D_s}(a) \downarrow$
- (ii) $u_{e,s}(a) < \max(0, \theta_s(a))$

If no such number a exists, the strategy does nothing. Otherwise it puts a restraint on a and *commits* to assigning $f_s(a) = \Phi_{e,s}^{D_s}(a)$. For any such a , this commitment will remain active as long as the strategy has a restraint on this element. Having done all this, the local strategy is declared to be satisfied and will not act again unless either a higher priority puts restraints on a , or the value of $u_{e,s}(a)$ or $\theta_s(a)$ changes. In both cases the strategy gets *injured* and has to reset, releasing all its restraints.

To finish stage s , the global strategy assigns values $f_s(y)$ for all $y \leq s$ as follows: if y is committed to some value assignment of $f_s(y)$ due to a local strategy, then define $f_s(y)$ to be this value. If not, let $f_s(y) = 0$. This finishes the construction and we now turn to the verification.

For each $e, a \in \omega$, let $Z_{e,a} = \{s \in \omega : \mathcal{R}_e \text{ restrains } a \text{ at stage } s\}$.

Claim. For each $e, a \in \omega$,

- (a) if $\Phi_e^D(a) \uparrow$ then $Z_{e,a}$ is finite;
- (b) if $\Phi_e^D(a) \downarrow = 1$ then $Z_{e,a}$ is either finite or cofinite.

Proof. By induction on the priority order. We consider $Z_{e,a}$, assuming that for all $\mathcal{R}_{e'}$ of higher priority, the set $Z_{e',a}$ is either finite or cofinite. First notice that $Z_{e,a} = \emptyset$ if $a < e$ or $a \notin \theta'$, so we may assume that $a \geq e$ and $a \in \theta'$. If there exists $e' < e$ such that $Z_{e',a}$ is cofinite, then $Z_{e,a}$ is finite because at most one requirement may claim a at a given stage. Suppose that $Z_{e',a}$ is finite for all $e' < e$. Fix t_0 such that for all $e' < e$ and $s \geq t_0$ $\mathcal{R}_{e'}$ does not restrain a at stage s . and $\theta_s(a) = \theta(a)$.

Suppose that $\Phi_e^D(a) \uparrow$. Fix $t_1 \geq t_0$ such that $D(b) = D_s(b)$ for all $b \leq \theta(a)$ and all $s \geq t_1$. Then for all $s \geq t_1$, if $\Phi_{e,s}^{D_s}(a) \downarrow$ then we must have $u_{e,s}(a) > \theta(a)$ because otherwise $\Phi_e^D(a) \downarrow$. It follows that for all $s \geq t_1$, requirement \mathcal{R}_e does not restrain a at stage s . Hence $Z_{e,a}$ is finite.

Suppose now that $\Phi_e^D(a) \downarrow$. Fix $t_1 \geq t_0$ such that for all $s \geq t_1$ we have $\Phi_{e,s}^{D_s}(a) \downarrow$ and $D_s(c) = D(c)$ for every $c \leq u_e(a)$. For every $s \geq t_1$, $u_{e,s}(a) = u_{e,t_1}(a)$ and $\theta_s(a) = \theta_{t_1}(a)$ for each $i \leq a$. So properties (i) and (ii) will either hold at each stage $s \geq t_1$, or not hold at each stage $s \geq t_1$. Therefore $Z_{e,a}$ is either finite or cofinite. \square

Claim. Each requirement \mathcal{R}_e is satisfied.

Proof. Suppose that Φ_e^D is total for some $e \in \omega$. We will prove that Φ_e^D is not an f -diagonalizing function. Let $A = \{a \geq e : (\forall e' < e) Z_{e',a} \text{ is finite}\}$. Notice that A is cofinite since for each $e' < e$, there is at most one a such that $Z_{e',a}$ is cofinite. Define $h : \omega \rightarrow \omega$ as follows.

If for all but finitely many $k \in \omega$, we have $k \in \theta' \rightarrow k \in \theta'_{u_e(k)}$, then $\theta' \leq_T u_e \leq_T D$, contrary to hypothesis. Thus we may let a be the least element of $\{k \in A : k \in \theta' \setminus \theta'_{u_e(k)}\}$ greater than e . We then have

- (1) $a \geq e$, $\Phi_e^D(a) \downarrow$, $\theta(a) > u_e(a)$
- (2) For all $e' < e$, there exists t such that $\mathcal{R}_{e'}$ does not claim a at any stage $s \geq t$.

Therefore we may fix $t \geq a$ such that for all $s \geq t$, we have $\Phi_{e,s}^{D_s}(a) \downarrow$, $\theta_s(a) = \theta(a)$, $u_{e,s}(a) = u_e(a)$, and for each $e' < e$, $\mathcal{R}_{e'}$ does not claim a at stage s . Thus, for every $s \geq t$, requirement \mathcal{R}_e claims $a' \leq a$ at stage s . Since $Z_{e,i}$ is either finite or cofinite for each $i \leq a$, it follows that $Z_{e,a}$ is cofinite. By the above argument, we must have $\Phi_e^D(a) \downarrow$, and by construction, $f(a) = \Phi_e^D(a)$. Therefore \mathcal{R}_e is satisfied. \square

Claim. The resulting function f_s is Δ_2^0 .

Proof. Consider a particular element a . Because of Claim 1, if $e > a$ then $Z_{e,a} = \emptyset$. We have then two cases: Either $Z_{e,a}$ is finite for all $e \leq a$, in which case for all but finitely many s , $f_s(a) = 0$, or $Z_{e,a}$ is cofinite for some e . Then there is a stage s at which requirement \mathcal{R}_e has committed $f_s(a) = \Phi_e^D(a)$ for assignment and has never been injured. Thus f is Δ_2^0 . \square

\square

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